This question paper contains 6 printed pages.

Your Roll No. ....

Sl. No. of Ques. Paper : **8508** **HC**

Unique Paper Code : **32357501**

Name of Paper : **Numerical Methods**

Name of Course : **Mathematics : DSE for Honours**

Semester : **V**

Duration : **3 hours**

Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Use of non-programmable scientific calculator is allowed.

*Attempt **all** questions, selecting **two** parts from each question.*

1. (a) Give the geometrical construction of the Newton's method to approximate a zero of a function. Write an algorithm to find a root of $f(x) = 0$ by Newton's method.

(b) Define order of convergence of an iterative scheme $\{x_n\}$. Determine the order of convergence for the recursive scheme:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

Turn over

(c) Define the rate of convergence of an iterative scheme $\{x_n\}$. Use the bisection method to determine the root of the equation $x^5 + 2x - 1 = 0$ on $(0, 1)$. Further, compute the theoretical error bound at the end of fifth iteration and the next enclosing (bracketing) interval. 13

2. (a) Differentiate between the method of false position and the secant method. Apply the method of false position to $\cos x - x = 0$ to determine an approximation to the root lying in the interval $(0, 1)$ until the absolute error is less than 10^{-3} $(p = 0{\cdot}739085)$.

(b) Let g be a continuous function on the closed interval $[a, b]$ with $g : [a, b] \to [a, b]$. Furthermore, suppose that g is differentiable on the open interval (a, b) and there exists a positive constant $k < 1$ such that $|g'(x)| \leq k < 1$ for all x belongs to (a, b). Then:

(i) The sequence $\{p_n\}$ generated by $p_n = g(p_n - 1)$ converges to the fixed point p for any p_0 belonging to $[a, b]$;

(ii) $|p_n - p_{n-1}| \leq k^n \max (p_0 - a, b - p_0)$.

(c) Find the approximated root of $f(x) = x^3 + 2x^2 - 3x - 1$ by secant method, taking $p_0 = 2$ and $p_1 = 1$ until $|p_n - p_{n-1}| < 5 \times 10^{-3}$. 13

3. (a) Using scaled partial pivoting during the factor step, find matrices L, U and P such that $LU = PA$ where

$$A = \begin{bmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{bmatrix}.$$

Hence, solve the system $Ax = b$, given

$$b = \begin{bmatrix} 14 \\ 36 \\ 7 \end{bmatrix}.$$

(b) Use Jacobi method to solve the following system of linear equations. Use the initial approximation $x^{(0)} = 0$ and perform three iterations.

$$4x_1 + 2x_2 - x_3 = 1$$
$$2x_1 + 4x_2 + x_3 = -1,$$
$$-x_1 + x_2 + 4x_3 = 1.$$

(c) (i) Consider the matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix}.$$

Find a lower triangular matrix L and an upper triangular matrix U with ones along its diagonal such that $A = LU$.

(ii) Determine the spectral radius of the matrix:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}.$$ 13

4. (a) (i) If $x_0, x_1, x_2, \ldots\ldots x_{n+1}$ are $n + 1$ distinct points and f is defined at $x_0, x_1, x_2, \ldots\ldots x_n$, then prove that interpolating polynomial P, of degree at most n, is unique.

(ii) Define the shift operator E and central difference operator δ. Prove that:

$$E = 1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}.$$

(b) For the function $f(x) = e^x$, construct the Lagrange form of interpolating polynomial of f passing through the points $(-1, e^{-1})$, $(0, 1)$ and $(1, e)$. Estimate $\sqrt{e}$ using the polynomial. What is the error in the approximation? Verify that theoretical error bound is satisfied.

(c) (i) Write the following data in the usual divided difference tabular form and determine the missing values:

$$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3,$$
$$f[x_0] = 2,\ f[x_1] = 6,\ f[x_2] = 6,$$
$$f[x_0, x_1] = 4,\ f[x_2, x_3] = 0,\ f[x_1, x_2, x_3] = 0.$$

(ii) Prove that:

$$\sum_{k=0}^{n-1} \Delta^2 f_k = \Delta f_n - \Delta f_0.$$ 12

5. (a) Use the formula:

$$f'(x) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

to approximate the derivative of the function $f(x) = 1 + x + x^3$ at $x_0 = 1$, taking $h = 1, 0.1, 0.01$, and 0.001. What is the order of approximation?

(b) Verify:

$$f'(x) \approx \frac{-3f(x_0)+4f(\pm h)-f(x_0+2h)}{2h},$$

the difference approximation for the first derivative provides the exact value of the derivative regardless of h, for the functions $f(x) = 1$, $f(x) = x$ and $f(x) = x^2$, but not for the function $f(x) = x^3$.

(c) Derive second-order forward difference approximation to the first order derivative of a function. 12

6. (a) Approximate the value of the integral $\int_1^2 \frac{1}{2} dx$ using Simpson rule. Further verify the theoretical error bound.

(b) Apply Euler's method to approximate the solution of the given initial value problem $x' + \frac{4}{t} = t^4$, $(1 \le t \le 3)$, $x(1) = 1$, $N = 5$. Further it is given that the exact solution is $x(t) = \frac{1}{9}(t^5 + 8t^{-4})$. Compute the absolute error at each step.

(c) Consider the initial value problem

$$x' = 1 + \frac{x}{t}, (1 \le t \le 3), x(1) = 1$$

whose exact solution is given by $x(t) = t(1 + \ln t)$. Using the step-size of 0.5, obtain the solution of the IVP and compare the absolute error with theoretical error bound, assuming the Lipschitz constant L equals 1. 12